UNIT-2

INTERPOLATION

Given a set of tabulated values (x_i, y_i) $i=0,1,2,\dots n$ ~~sto~~ satisfying the relation $y=f(x)$ where the explicit nature of $f(x)$ is not known it is required to find a simpler function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agrees at the set of tabulated point such a process is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation. If $\phi(x)$ is called interpolating polynomial. Similarly diff types of interpolation arise depending on whether $\phi(x)$ is finite trignometric series, series of basic f^n.

Here we shall be concerned with polynomial interpolation only.

Newton Forward and Backward Interpolation Formula

Given $(n+1)$ points say (x_i, y_i), $i=0,1,2,3,\dots n$

Such that $x_{i+1} = x_0 + ih$, it is required to interpolate

$y(x_0+ph) = y_p$ (say),

where, p is real no.

value at which you want to interpolate the data

first entry of argument

$x = x_0 + ph$

$p = \frac{x - x_0}{h}$

$$NFIF \cong y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

Last entry of given data

$$NBIF \cong y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \dots$$

$p = \frac{x - x_n}{h}$ $h \rightarrow$ interval of differencing (in this case)

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Remark: NFIF is useful for interpolating the values near the beginning of the table whereas NBIF is useful for interpolating the values near the end of the table.

Derivation for NFIF & NBIF :—

Given that $(x_i, y_i), i = 0, 1, 2, \ldots n$ such that x_i are equally spaced, let $x = x_0 + ph$, then

$$f(x) = y(x) = y(x_0 + ph) = E^p y(x_0)$$
$$= (1+\Delta)^p y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots$$

which is known as NFIF.

Similarly, put $x = x_n + ph$

$$y(x) = y(x_n + ph) = E^p y(x_n) = (E^{-1})^{-p} y(x_n)$$
$$= (1-\nabla)^{-p} y_n$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

Q. Find the interpolating polynomial which passes through the point $(0,5), (1,3), (2,3), (3,5)$ $\quad h=1$

$h = 1$ in every ~~term~~ ⇒ x_i are equally spaced.

x	y	$\Delta f(a)$	$\Delta^2 f(a)$	$\Delta^3 f(a)$
0	5			
		-2 (Δy_0)		
1	3		2 $(\Delta^2 y_0)$	
		0		0
2	3		2 $\Delta^2 y_n$	
		2 Δy_n		
3	5			

By using the forward & backward interpolating formula

For Forward

$$p = \frac{x - x_0}{h}$$

$x_0 = 0$, $h = 1$

$x_0 = 0$, $h = 1$

$\boxed{p = x}$

NFIF

$$f(x) = y = 5 + x(-2) + \frac{\cancel{x}(\cancel{x}-1)}{\cancel{2}} \cancel{\nabla^2 y_0} (\cancel{2})$$

$$\boxed{f(x) = x^2 - 3x + 5}$$

NBIF

$$f(x) = y_n = 5 + 2(x-3) + \frac{(x-3)(x-2)}{\cancel{2!}} \times \cancel{2} \quad \{x_n = 3$$

$$= 5 + 2x - \cancel{6} + x^2 - 5x + \cancel{6}$$

$$\boxed{f(x) = x^2 - 3x + 5}$$

Q) Use some suitable interpolation formula to compute the value of y at :

$x = 6, 6.25, 6.45, 7.3, 7.5$ and 7.8

(6.2, 238.328)
(6.4, 262.144)
(6.6, 287.494)
(6.8, 314.432)
(7.0, 343.0)
(7.2, 373.248)
(7.4, 405.224)
(7.6, 438.976)

$h = 0.2$ for all values
⇒ NFIF & NBIF will be used
Since, the values 6, 6.25 & 6.45 are near the beginning so NFIF will used
and the values 7.3, 7.5 & 7.8 are near the end so NBIF will be used.

x	y	$\Delta f(a)$	$\Delta^2 f(a)$	$\Delta^3 f(a)$
6.2	238.328	23.816 Δy_0	1.534 $\Delta^2 y_0$	0.054 $\Delta^3 y_0$
6.4	262.144	25.35	1.588	0.042
6.6	287.496	26.938	1.63	0.05
6.8	314.432	28.568	1.68	0.048
7.0	343.0	30.248	1.728	0.048
7.2	373.248	31.976	1.776	
7.4	405.224	33.752		
7.6	438.976			

$\Delta^4 f(a)$	$\Delta^5 f(a)$	$\Delta^6 f(a)$	$\Delta^7 f(a)$
-0.012 $\Delta^4 y_0$	0.02 $\Delta^5 y_0$	-0.01 $\Delta^6 y_0$	-0.002 $\Delta^7 y_0$
0.008	0.01	-0.008	
-0.002	0.002		
0			

$h = 0.2$

$x_0 = 6.2$

$p = \frac{x - x_0}{h}$

$p = \left(\frac{x - 6.2}{0.2}\right)$

$f(x) = y = 238.328 + \left(\frac{x-6.2}{0.2}\right)(23.816) + \left(\frac{x-6.2}{0.2}\right)\left(\frac{x-6.4}{0.2}\right)\frac{1.534}{2!}$

$+ \left(\frac{x-6.2}{0.2}\right)\left(\frac{x-6.4}{0.2}\right)\left(\frac{x-6.6}{0.2}\right)\left(\frac{0.054}{3!}\right) + \left(\frac{x-6.2}{0.2}\right)\left(\frac{x-6.4}{0.2}\right)\left(\frac{x-6.6}{0.2}\right)$

$\left(\frac{x-6.8}{0.2}\right)(-0.012) + \left(\frac{x-6.2}{0.2}\right)\left(\frac{x-6.4}{0.2}\right)\left(\frac{x-6.6}{0.2}\right)\left(\frac{x-6.8}{0.2}\right)$

$\left(\frac{x-7}{0.2}\right)(0.02) + \left(\frac{x-6.2}{0.2}\right)\left(x-6.4\right.$

Q) The following table is the population of the town during the last 6 census. Estimate by using Newton interpolation the inc. in population during 1946 – 1948.

year :	1911	1921	1931	1941	1951	1961
Population: (in Thousand)	12	15	20	27	39	52

year x	Population y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1911	12					
		3				
1921	15		2			
		5		0		
1931	20		2		+3	
		7		3		−10 $\nabla^5 y_n$
1941	27		5		−7 $\nabla^4 y_n$	
		12		−4 $\nabla^3 y_n$		
1951	39		1 $\nabla^2 y_n$			
		13 ∇y_n				
1961	52 y_n					

NBIF

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cancel{\frac{p(p+1)(p+2)(p+}{4!}\nabla^4 y_n}$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)\ldots(p+4)}{5!}\nabla^5 y_n$$

I) For the year 1946

$$p = \frac{x - x_n}{h}$$

$$= \frac{1946-1961}{10} = \frac{-15}{10} = -1.5$$

II For the year 1948

$$p = \frac{x - x_n}{h} = \frac{1948-1961}{10} = \frac{-13}{10} = -1.3$$

Q) Using ~~Newton forward~~ NFIF ~~interp~~ find y at $x=8$ from the following table:-

x :	0	5 *	10	15	20	25
y :	7	11	14	18	24	32

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y_0$
0	7 y_0	4 Δy_0	-1 $\Delta^2 y_0$	2 $\Delta^3 y_0$	-1 $\Delta^4 y_0$	0 $\Delta^5 y_0$
5	11	3	1	1	-1	
10	14	4	2	0		
15	18	6	2			
20	24	8				
25	32					

NFIF

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_n + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_n + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_n$$

$$p = \frac{x-x_0}{h} = \frac{8-0}{5} = \frac{8}{5}$$

$$y(8) = 7 + \frac{8}{5}(4) + \frac{\left(\frac{8}{5}\right)\left(\frac{8}{5}-1\right)}{2!}(-1) + \frac{\left(\frac{8}{5}\right)\left(\frac{8}{5}-1\right)\left(\frac{8}{5}-2\right)}{3!}(2) +$$

$$\frac{\left(\frac{8}{5}\right)\left(\frac{8}{5}-1\right)\left(\frac{8}{5}-2\right)\left(\frac{8}{5}-3\right)}{4!}(-1) + \frac{\left(\frac{8}{5}\right)\cdots\left(\frac{8}{5}-4\right)}{5!}(0)$$

$$= 7 + \frac{32}{5} +$$

Divided Difference

Let $y = f(x)$ takes the value $f(x_0), f(x_1), f(x_2), \ldots f(x_n)$ ($\downarrow y_0, y_1, y_2 \ldots y_n$)

corresponding to the values $x_0, x_1, x_2 \ldots x_n$ respectively where x_i are not equally spaced. We define the first divide diff. of $f(x)$ with arguments (x_0, x_1)

$$f(x_0, x_1) = [x_0, x_1] = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$f(x_1, x_2) = [x_1, x_2] = \underset{x_2}{\Delta} f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

The second divided diff. of $f(x)$ for 3 arguments say x_0, x_1, x_2 is defined as -

$$f(x_0, x_1, x_2) = [x_0, x_1, x_2] = \underset{x_1; x_2}{\Delta} f(x_0) = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2)}{x_3 - x_0}$$

Some properties of divided difference:-

1. The divided diff are symmetrical in all their arguments. i.e. $[x_0, x_1] = [x_1, x_0]$,

$$[x_0, x_1, x_2] = [x_1, x_2, x_0]$$

2. The divided diff satisfies the linearity property

~~3. Divided diff of~~

3. n^{th} divided diff $\{\Delta^n f(x)\}$ is const if $f(x)$ is a polynomial of degree n.

Q-4 Construct the divided diff. table for the following data:

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	3			
		$\frac{-6-3}{0-(-1)} = -9$		
0	-6		$\frac{15-(-9)}{3-(-1)} = 6$	
		$\frac{39-(-6)}{3-0} = 15$		
3	39		$\frac{261-15}{6-0} = 41$	
		$\frac{822-39}{6-3} = 261$		
6	822		$\frac{339-261}{7-3} =$	
		$\frac{1161-822}{7-6} = 339$		
7	1161			

Construct divided diff table for the f^n, $f(x) = x^x$ with the arguments $x = -2, 1, 3, 4$

x	$y(x^x)$	$\Delta f(x)$	$\Delta^2 f(x)$
-2	0.25		
		$\frac{1-0.25}{1+2} = 0.25$	
1	1		$\frac{13-0.25}{3-(-2)}$
		$\frac{27-1}{3-1} = 13$	
3	27		
		$\frac{256-27}{1} = 229$	
4	256		

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Newton General Interpolation formula or Newton Divided Difference Interpolation formula -

Given (x_i, y_i), $i = 0, 1, 2, \ldots n$ such that x_i are not equally spaced. By the defn of divided difference

$$\underset{[x, x_0]}{f(x, x_0)} = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0) f(x, x_0)$$

Again, consider the second diff.

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\Rightarrow f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$$

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1)$$

Proceeding in this way we get here,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1) + \ldots + (x - x_0)(x - x_1) \ldots (x - x_n) f(x, x_0, x_1 \ldots x_n)$$

Q. Use NDDIF to interpolate the polynomial which passes through the point $(-1, 3)$, $(0, -6)$, $(3, 39)$, $(6, 822)$, $(7, 1611)$

Q. Prove that the third divided difference with arguments a, b, c, d of the function $f(x) = \frac{1}{x}$ is equal to $\frac{-1}{abcd}$

Let, $x_0 = a$, $x_1 = b$, $x_2 = c$, $x_3 = d$

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab}$$

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$$f(x_0,x_1,x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0} = \frac{\frac{-1}{bc}-\left(\frac{-1}{ab}\right)}{c-a}$$

$$f(x_0,x_1,x_2) = \frac{1}{abc}$$

$$f(x_0,x_1,x_2,x_3) = \frac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{x_3-x_0}$$

$$= \frac{\frac{1}{bcd}-\frac{1}{abc}}{d-a}$$

$$f(x_0,x_1,x_2,x_3) = \frac{-1}{abcd}$$

Lagrange Interpolation Formula :-

Given the points (x_i, y_1), $i=0,1,3,\dots n$ where x_i are not ~~ee~~ necessarily equally spaced then we can find the n^{th} degree interpolating polynomial by using the formula

$$y(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}y_0 +$$

$$\frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_0-x_n)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}y_2$$

$$\dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}y_n$$

which is known as LIF. and this method is useful to interpolate any value in the interval $x \in [x_0, x_n]$ OR $x_0 \le x \le x_n$

Q11) Interpolate y at $x = 1.7$ by LIF from the following table :-

	x_0	x_1	x_2	x_3
x :	1.2	1.5	1.9	2.3
y :	-2.472	2.625	12.09	25.017
	y_0	y_1	y_2	y_3

$y(x) = 6.963$

$x = 1.7$

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Q) ~~Interp~~ Using LIF find the interpolating polynomial satisfying the given data in Q.1

Q) Using LIF express x as a polynomial f^n of y that fits the following data.

$x = f(y)$

x:	0	2	3
y:	0	4	9

Q) Use LIF to approximate $f(x) = \log_{10} 301$ correct to 4 decimal places from the table given below.

x:	300	304	305	307
y: $y = \log_{10} x$:	2.4771	2.4829	2.4843	2.4871

Q) Using LIF find x when $\sqrt[3]{x} = 3.756$ from the given table

x:	50	52	54	56
$y = \sqrt[3]{x}$:	3.684	3.732	3.779	3.825

NUMERICAL DIFFERENTIATION

Consider a set of regular values $(x_i, y_i), i = 0, 1, 2 \ldots n$ such that x_i are equally spaced i.e. $x_{i+1} = x_0 + ih$. It is required to find $y', y'', y''' \ldots y^{(n)}$ for any $x \in [x_0, x_n]$

The General method for numerical diff^n is to differentiate the given interpolating polynomial. Hence different formulas for diff^n can be obtained corresponding to different interpolating formula.

Consider NFIF

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots \ldots + \frac{p(p-1)(p-2)\ldots\{p-(n-1)\}}{n!}\Delta^n y_0 \quad (1)$$

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where $p = \frac{x + x_0}{h}$

Now diff (1) w.r.t. x

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \frac{5p^4-40p^3+105p^2-100p+24}{5!}\Delta + \cdots\right]$$